M^2=Math Mediator — Lesson 25: Solve Systems of Equations Using Matrices

Total Recall (Warm-up) (5 minutes approx.)

Total Recall: Exercise from yesterday's lesson on Matrix inverses and Determinants.

1. What is the determinant of this Matrix? $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$ Ans: $3 \cdot 6 - 4 \cdot 5 = -2$

2. Find the determinant: $\begin{vmatrix} 1 & 4 & 1 \\ 2 & 5 & 2 \\ 3 & 6 & 3 \end{vmatrix}$ (notice the different enclosure lines?)

   Answer: $(1 \cdot 5 \cdot 3 + 4 \cdot 2 \cdot 3 + 1 \cdot 2 \cdot 6) - (3 \cdot 5 \cdot 1 + 6 \cdot 2 \cdot 1 + 3 \cdot 2 \cdot 4) = (15 + 24 + 12) - (15 + 12 + 24) = 0$.

3. Given that the inverse of a matrix is $\frac{1}{\text{det}}$ times matrix of the coefficients; why would the determinant in #2 above result in no solution for the inverse of a matrix? Answer: because 1/0 is undefined.

Direct Instruction (15 minutes approx.)

Emissions of carbon dioxide or CO_2 is of great concern among factories and environmental agencies. In one theoretical city there were 6 factories that the EPA (environmental protective agency) was concerned with emissions and might have used matrices to analyze their emissions.

Matrix: Fuel Quantities used by factories · Matrix: CO_2 per fuel = Matrix: Total CO_2 per factory

The EPA wished to analyze the amount of CO_2 per fuel was generated by the factories. They measured the total CO_2 output per factory and received a list of the types of fuels used by each factory and used matrix multiplication to solve for the unknowns. The below matrix shows FactoryA using 10 tons of fuel 1 (gasoline perhaps), 2.5 tons of fuel 2 (maybe coal), 7 tons of fuel 3, 0.5 tons of fuel 4, 0 tons of fuel 5 and 1.3 tons of fuel 6.

Fuel Quantities by factory — CO_2 per fuel — Total CO_2

$$\begin{matrix} FactoryA \\ FactoryB \\ FactoryC \\ FactoryD \\ FactoryE \\ FactoryF \end{matrix} \begin{bmatrix} 10 & 2.5 & 7 & 0.5 & 0 & 1.3 \\ 5 & 7 & 3 & 0 & 2 & 2 \\ 20 & 30 & 5 & 2 & 1.2 & 4 \\ 0.5 & 3 & 8 & 10 & 1.5 & 0.3 \\ 3 & 6 & 15 & 8 & 0.1 & 3.4 \\ 4 & 18 & 2 & 6 & 4 & 2 \end{bmatrix} \text{x} \begin{bmatrix} g \\ h \\ j \\ k \\ m \\ n \end{bmatrix} = \begin{bmatrix} 302.6 \\ 246 \\ 916.6 \\ 224.1 \\ 339.6 \\ 450 \end{bmatrix}$$

For FactoryA: $10g + 2.5h + 7j + 0.5k + 0m + 1.3n = 302.6$ and there are six factories with similar equations and the EPA wants to solve for the 6 variables.

We could use inverse matrices and determinants to manually solve this equation, and people did that before computers and calculators. Let's use the calculator to solve this complex problem. Enter into the calculator the factory fuel usage

	matrix as Matrix A and the total CO_2 matrix as Matrix B. Use the matrix math function on your calculator to solve for the unknowns by multiplying the inverse of Matrix A by Matrix B. The solution is: $\begin{bmatrix} 22 \\ 13 \\ 5 \\ 12 \\ 6 \\ 7 \end{bmatrix}$
Review (5 minutes approx.)	U-DO: Here is another example that you can try on your calculator: There is a video game review that scores video games on graphics, violence, action and cost. Each of these attributes are scored and are weighted and combined to give total scores for each video game. Lets use the information to create our own criteria. Given the review ratings matrix, we will multiply it by our weighing criteria to determine total scores and see which game might be best for the chosen criteria. Video Game Review Ratings $\begin{matrix} VideoGame1 \\ VideoGame2 \\ VideoGame3 \\ VideoGame4 \end{matrix} \begin{bmatrix} 8 & 7 & 7 & 3 \\ 3 & 4 & 4 & 5 \\ 9 & 8 & 8 & 9 \\ 4 & 2 & 5 & 6 \end{bmatrix}$ These 4 video games are rated for graphics, violence, action and cost, which are the columns of the matrix. We will create a weighing matrix based on our desire for great graphics and action (10 each), violence isn't too much of a concern (1); and cost is no object (-5). We now have the matrices and we will multiply them to find the best product: $\begin{matrix} VideoGame1 \\ VideoGame2 \\ VideoGame3 \\ VideoGame4 \end{matrix} \begin{bmatrix} 8 & 7 & 7 & 3 \\ 3 & 4 & 4 & 5 \\ 9 & 8 & 8 & 9 \\ 4 & 2 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 1 \\ 10 \\ -5 \end{bmatrix} = ?$ Using the calculator: $\begin{bmatrix} 142 \\ 49 \\ 133 \\ 62 \end{bmatrix} \begin{matrix} VideoGame1 \\ VideoGame2 \\ VideoGame3 \\ VideoGame4 \end{matrix}$ Video game 1 is our best choice. Graphing Calculators make short work of systems of equations using matrices.
Review (5 minutes approx.)	Here is a short summary of methods of solving systems of equations: 1. Elimination: Use this non-calculator method for 2 variables where the coefficients are not 1 or -1. Set the coefficients on one variable the same or equal and opposite and subtract or add the equations. 2. Substitution: Use this non-calculator method for 2 variables where the coefficients are 1 or -1. Solve one equations for the variable with the 1, -1 coefficient and substitute into the other equation.

	3. Matrices: Use this if a calculator is handy and allowed. 4. Graphing the solution: Use this if the linear equations are easy to graph. That is if they are easily put into the y = mx + b form.
Practice (15 minutes approx.)	Try these examples from the summary: 1. Solve by substitution: 3x + y = 7 and 4x + 2y = 16. Answer: (-1, 10) 2. Solve by Graphing: y = 2x -1 and y = (x/2) + 2. Answer: (2, 3) 3. Solve by elimination: 3x + 2y = 4 and 5x + 2y = 16. Answer: (6, -7) 4. Solve with Matrices: 3x + 6y = -15 and 5x + 9y = 6 a. $\begin{bmatrix} 3 & 6 \\ 5 & 9 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -15 \\ 6 \end{bmatrix}$ or $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 5 & 9 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -15 \\ 6 \end{bmatrix}$ b. Or $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\det}\begin{bmatrix} 9 & -6 \\ -5 & 3 \end{bmatrix} \cdot \begin{bmatrix} -15 \\ 6 \end{bmatrix}$ because inv = $\frac{1}{\det}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ c. And det = 3·9-5·6 = -3 and $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-3}\begin{bmatrix} 9 & -6 \\ -5 & 3 \end{bmatrix} \cdot \begin{bmatrix} -15 \\ 6 \end{bmatrix}$ d. Or taking 1/3 of the 2x1 matrix: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 & -6 \\ -5 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ e. And then x = 9·5 + (-6)·(-2) = 45 + 12 = 57 f. And y = -5·5 + 3·(-2) = -25 – 6 = -31
Exercise: (10 minutes approx.)	Use Cramer's Rule to solve for the following systems of equations: 5x + 4y = 5 and 9x – 8y = 0 $x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\det}; y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\det}$; where ax + by = e and cx + dy = f. Answer: $x = \frac{\begin{vmatrix} 5 & 4 \\ 0 & -8 \end{vmatrix}}{\begin{vmatrix} 5 & 4 \\ 9 & -8 \end{vmatrix}}; y = \frac{\begin{vmatrix} 5 & 5 \\ 9 & 0 \end{vmatrix}}{\begin{vmatrix} 5 & 4 \\ 9 & -8 \end{vmatrix}}$; x = $\frac{5\cdot(-8) - 0\cdot 4}{5\cdot -8 - 9\cdot 4} = \frac{-40}{-76}$; y = $\frac{5\cdot 0 - 9\cdot 5}{-76}$ x = 10/19; y = 45/76
Wrap-up (5 minutes approx.)	Wrap up closing comments and housekeeping.