M^2=Math Mediator   Lesson 24: Determinants

Intro:
(5 minutes approx.)

Total Recall: Exercise from yesterday's lesson on Matrix Multiplication.

1. Does the multiplication of matrices M_{1x3} and N_{3x2} have a solution? Why or why not? What is the resultant matrix dimension if there is a solution?
   Answer: Yes, columns M match rows N, the resultant is 1 x 2.
   Remember in matrix dimension, rows are first, columns second. (RC cola).
2. How about multiplying matrices R_{4x2} and S_{4x1}; same questions as in #1?
   Answer: No solution because the columns of R do not match rows S.
3. How about multiplying matrices A_{3x2} and B_{3x2}; same questions as in #1?
   Answer: No solution because the columns of A do not match rows B.
4. What is the product: $\begin{bmatrix} 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = ?$ The answer is [3x1 + 2x2] = [7]

1. $\begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix} = ?$ $\begin{bmatrix} 1 \cdot 2 + 4 \cdot (-3) \\ (-1) \cdot 2 + 2 \cdot (-3) \end{bmatrix} = \begin{bmatrix} -10 \\ -8 \end{bmatrix}$

Direct Instruction
(10 minutes approx.)

Matrices are used in breeding horses. Champion horse owners want to know what the outcome will be for breeding horses. Genetics research people have identified genetic traits and combinations to predict successful horse breeding. Some breeders have used much more than 10 traits, but we will only use 10 for demonstration purposes today. Our situation is that there are 10 stallions that we want to breed with one mare. What will be the combination of their genes?

Traits / Horse	1	2	3	4	5	6	7	8	9	10
A	2	0	3	-4	0	1	6	-2	0	4
B	4	2	-2	-2	6	-2	-1	6	5	2
C	5	2	1	0	0	3	6	-2	-3	1
D	1	0	2	-2	1	0	5	0	2	2
E	3	1	3	0	3	2	-2	3	2	2
F	4	3	0	-1	1	1	0	2	3	0
G	3	3	1	-2	4	2	2	0	2	3
H	0	3	-3	0	2	4	-2	5	3	1
I	3	1	4	-3	-1	3	4	1	0	2
J	-2	-1	5	4	1	3	5	-1	1	5

Mare's Traits	
1	0
2	-2
3	3
4	5
5	0
6	1
7	5
8	2
9	-3
10	2

The matrix information is in the bold area.

Multiplying the 10x10 matrix by the 10x1 results in a 10x1 matrix with 10 possible combinations of genes per mating.

$$\begin{bmatrix} 2 & 0 & 3 & -4 & 0 & 1 & 6 & -2 & 0 & 4 \\ 4 & 2 & -2 & -2 & 6 & -2 & -1 & 6 & 5 & 2 \\ 5 & 2 & 1 & 0 & 0 & 3 & 6 & -2 & -3 & 1 \\ 1 & 0 & 2 & -2 & 1 & 0 & 5 & 0 & 2 & 2 \\ 3 & 1 & 3 & 0 & 3 & 2 & -2 & 3 & 2 & 2 \\ 4 & 3 & 0 & -1 & 1 & 1 & 0 & 2 & 3 & 0 \\ 3 & 3 & 1 & -2 & 4 & 2 & 2 & 0 & 2 & 3 \\ 0 & 3 & -3 & 0 & 2 & 4 & -2 & 5 & 3 & 1 \\ 3 & 1 & 4 & -3 & -1 & 3 & 4 & 1 & 0 & 2 \\ -2 & -1 & 5 & 4 & 1 & 3 & 5 & -1 & 1 & 5 \end{bmatrix} \times \begin{bmatrix} 0 \\ -2 \\ 3 \\ 5 \\ 0 \\ 1 \\ 5 \\ 2 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} AP \\ BP \\ CP \\ DP \\ EP \\ FP \\ GP \\ HP \\ IP \\ JP \end{bmatrix}$$

Where AP, BP, CP and so on are a combination of the matching traits score. This is actually simplistic, because real genetic combinations are much more complex.

Other uses for matrices of this size are in engineering (multiplying electrical currents or mechanical forces), finances (multiplying stock or fund returns), and education (grading students assignments and weighing each type).

Taking the horse breeding situation, the breeder may want to match 10 stallions and 10 foal (baby horses) to one theoretical mare, and then try to find that mare or something close to the genetic makeup. Using the matrix multiplication above, if the genetics person could solve for the mare’s matrix by multiplying both sides of the multiplication by the inverse of the 10 stallion matrix, they would be very happy and could start their search for the mare. Multiplying by the inverse is the same as dividing by the original. Using the inverse is the method for matrix solutions.

Using ‘S’ for the 10 stallion matrix, ‘M’ for the mare and ‘R’ for the result:

S x M = R; and then (S^{-1}) x S x M = (S^{-1}) x R; and finally 1 x M = (S^{-1}) x R

We can solve for the mare’s matrix by multiplying the result matrix by the inverse matrix. In order to find the inverse matrix, we first need to know about DETERMINANTS, because the inverse of a matrix (S^{-1}) is equal to $\frac{1}{\det A}[Matrix : Coefficients\_of\_A]$.

Review (10 minutes approx.)

The determinant of a matrix must be a square matrix: 2x2 or 3x3 or 10x10 or such. If there is a matrix D = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$;

then the determinant of D or det D = $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$; notice the new marks, straight lines enclose the determinant, where bracketed lines enclosed the matrix.

The method to find the determinant D is to cross multiply and subtract the two

	products: det D = $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = ad – cb For a 3x3 matrix: E = $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$; det E = $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$; in order to cross multiply the 3x3 matrix, we need to extend the columns out by repeating the beginning columns after the right enclosure line: Det E = $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$ and then multiply the top down diagonals, add them and then subtract the bottom up diagonals: det E = (aei + bfg + cdh) – (gec + hfa + idb). Examples: $\det A = \begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix} = (4\cdot5)-(2\cdot3) = 20-6 = 14$ $\begin{vmatrix} 2 & 4 & 3 \\ 5 & 2 & 1 \\ 6 & 3 & 3 \end{vmatrix} = (2\cdot2\cdot3+4\cdot1\cdot6+3\cdot5\cdot3)-(6\cdot2\cdot3+3\cdot1\cdot2+3\cdot5\cdot4) = (12+24+45)$ - (36 + 6 + 60) = 81-102 = -21
Practice (12 minutes approx.)	U-DO: Find the determinants: 1. $\begin{vmatrix} 3 & 5 \\ 6 & 7 \end{vmatrix}$ Answer: (3·7)-(6·5) = 21 – 30 = -9 2. $\begin{vmatrix} 2 & 1 & 4 \\ 3 & 4 & 2 \\ 1 & 5 & 3 \end{vmatrix}$ Answer: (2·2·3+1·2·1+4·3·5) - (1·4·4+5·2·2+3·3·1) = (12+2+60) – (16+20+9) = 74-45 = 29 3. Verify your answers using the 'det' function on your calculators
Direct Instruction: (15 minutes approx.)	Now we will see how matrices are used to solve systems of equations. For the two equations: ax + by = e and cx + dy = f; we can remove the coefficients of the x and y variables and form the matrix multiplication equality: $\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\cdot\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$ because of what we have learned about matrix multiplication and equal matrices have to have corresponding elements equal. To solve for the values of x and y; we must take the inverse coefficient matrix and multiply it by both sides of the equality, just like regular multiplication solution.

If: M · X = N; then X = N/M or M^{-1} · N and with matrices: And remember that with matrices, order matters, so the inverse must come first, unlike regular multiplication!

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \cdot \begin{bmatrix} e \\ f \end{bmatrix}$; the definition of the inverse of a matrix is the reciprocal of the determinant of the matrix times a matrix of the coefficients re-organized:

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$; the re-organized coefficients matrix is defined and will always use the same organization with other equations and other coefficients.

U-DO: find $\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}^{-1}$ using the definition above. Answer:

$$\frac{1}{(2\cdot 8-3\cdot 5)}\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$$

Continuing: using the inverse matrix identity to solve systems of equations:

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \cdot \begin{bmatrix} e \\ f \end{bmatrix} = \frac{1}{\det}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cdot \begin{bmatrix} e \\ f \end{bmatrix} = \frac{1}{\det}\begin{bmatrix} de+(-b)f \\ (-c)e+af \end{bmatrix}$ which we can now use the fact that matrices multiplied by a constant is equal to each element times that constant. The constant here is 1/det. Equal matrices have equal elements so: $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\det}\begin{bmatrix} de+(-b)f \\ (-c)e+af \end{bmatrix}$ gives us x = $\frac{de-bf}{\det}$ and y = $\frac{af-ce}{\det}$

This can be restated as what is known as **Cramer's Rule**:

To solve for a system of two variable equations ax + by = e and cx + dy = f then

$$x = \frac{de-bf}{\det} = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\det} = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{(ed-fb)}{(ad-cb)}$$; and

$$y = \frac{af-ce}{\det} = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\det} = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & c \\ b & d \end{vmatrix}} = \frac{(af-ce)}{(ad-cb)}$$

If time: Example of using matrices to solve a system of equations: 4x + y = 10 and 3x + 5y = -1; define the variables a, b, c, d, e and f and create the matrix

	equation: $\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$ and $\begin{bmatrix} 4x+1y \\ 3x+5y \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \end{bmatrix}$
Wrap-up (3 minutes approx.)	Wrap up closing comments and housekeeping.